```
> restart;
```

This Maple worksheet compares P.G. Bergmann's notation in the paper:
Two-Component Spinors in General Relativity
Phys. Rev., v. 107, n. 2, p.624, 1956.
with the formalism of absolute parallelism as presented by G.I. Shipov in his book:
A Theory of Physical Vacuum
Russian Academy of Natural Sciences, 1998.

Bergmann's methods are employed by Mendel Sachs in his book:
General Relativity and Matter
Reidel Publishing Co., 1982.

Last revised: 2 Sep 2000.

Bookmarks:

- Two dimensional (x,t) case
- Christoffel symbols
- general spin-affine connection
- Ricci rotation coefficients
- Bergmann calculations
- spin curvature and Ricci curvature

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```
> read `bergmann4d7.m`; # restart from where we left off
  read `tensorlite.m`;
  constants:=op({constants} minus {gamma}):
  unprotect(gamma):
```

```
with(linalg):
linalg[transpose](0):=0: # tell them how to handle
linalg[htranspose](0):=0:
`print/transpose`:=proc(x) x^`T`; end: #transpose
`print/htranspose`:=proc(x) x^`+`; end: #Hermitean
```

```
Warning, new definition for norm
Warning, new definition for trace
```

Expand indices in expressions like phi(x[-i])

```
> All:=proc(x)
    if x::atomic then
       expand(Tensor(x))
    else
       map(All,x)
    fi
  end:
```

4 dimensional base manifold.

```
> dimension:=4:
> deadindex:={a,b,c,d,e,f,g,h}: # fibre indices
```

Co-ordinates

```
> let('x[-j]=
    [x1,x2,x3,x0]
  [j]'):
```

Orthonormal space-space components of the tetrad.

```
> let(qxy[-a,b]='All([
    [cos(phi12(x[-j])),sin(phi12(x[-j])),0,0],
    [-sin(phi12(x[-j])),cos(phi12(x[-j])),0,0],
    [0,0,1,0],
    [0,0,0,1]
  ][b,a])'):
> let(qxz[-a,b]='All([
    [cos(phi13(x[-j])),0,sin(phi13(x[-j])),0],
    [0,1,0,0],
    [-sin(phi13(x[-j])),0,cos(phi13(x[-j])),0],
    [0,0,0,1]
  ]
```

```
  [b,a])'):
> let(qyz[-a,b]='All([
    [1,0,0,0],
    [0,cos(phi23(x[-j])),sin(phi23(x[-j])),0],
    [0,-sin(phi23(x[-j])),cos(phi23(x[-j])),0],
    [0,0,0,1]
  ]
  [b,a])'):
```

Orthonormal space-time components of the tetrad.

```
> let(qxt[-a,b]='All([
    [cosh(phi14(x[-j])),0,0,-sinh(phi14(x[-j]))],
    [0,1,0,0],
    [0,0,1,0],
    [-sinh(phi14(x[-j])),0,0,cosh(phi14(x[-j]))]
  ]
  [b,a])'):
> let(qyt[-a,b]='All([
    [1,0,0,0],
    [0,cosh(phi24(x[-j])),0,-sinh(phi24(x[-j]))],
    [0,0,1,0],
    [0,-sinh(phi24(x[-j])),0,cosh(phi24(x[-j]))]
  ]
  [b,a])'):
> let(qzt[-a,b]='All([
    [1,0,0,0],
    [0,1,0,0],
    [0,0,cosh(phi34(x[-j])),-sinh(phi34(x[-j]))],
    [0,0,-sinh(phi34(x[-j])),cosh(phi34(x[-j]))]
  ]
  [b,a])'):
```

(Pseudo) orthogonal factor

```
> infolevel[let]:=0:
  let('q[-a,g]=qxt[-a,b]*qyt[-b,c]*qzt[-c,d]*qxy[-d,e]*qxz[-e,f]*
  qyz[-f,g]',simplify);
```

$$q_{-a,\,g} = qxt_{-a,\,b}\, qyt_{-b,\,c}\, qzt_{-c,\,d}\, qxy_{-d,\,e}\, qxz_{-e,\,f}\, qyz_{-f,\,g}$$

Symmetric factor

```
> infolevel[let]:=0:
  let(s[-a,j]='All([
  [s11(x[-k]),s12(x[-k]),s13(x[-k]),s14(x[-k])],
  [s12(x[-k]),s22(x[-k]),s23(x[-k]),s24(x[-k])],
  [s13(x[-k]),s23(x[-k]),s33(x[-k]),s34(x[-k])],
  [s14(x[-k]),s24(x[-k]),s34(x[-k]),s44(x[-k])]
  ][a,j])'):
> #matrix(4,4,(a,j)->s[-a,j]);
```

Two dimensional (x,t) general case

```
> eq1:=eval(All({
  phi12(x[-j])=0,phi13(x[-j])=0,phi23(x[-j])=0,
  phi14(x[-j])=phi14(x1,x0),
  phi24(x[-j])=0,phi34(x[-j])=0,
  s11(x[-j])=s11(x1,x0),
  s12(x[-j])=0,s13(x[-j])=0,
  s14(x[-j])=s14(x1,x0),
  s22(x[-j])=1,s23(x[-j])=0,s24(x[-j])=0,
  s33(x[-j])=1,s34(x[-j])=0,
  s44(x[-j])=s44(x1,x0)
  })):
> matrix(dimension$2,(a,b)->simplify(subs(eq1,q[-a,b])));
```

$$\begin{bmatrix} \cosh(\phi 14(x1,x0)) & 0 & 0 & -\sinh(\phi 14(x1,x0)) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\phi 14(x1,x0)) & 0 & 0 & \cosh(\phi 14(x1,x0)) \end{bmatrix}$$

```
> matrix(dimension$2,(a,j)->subs(eq1,s[-a,j]));
```

$$\begin{bmatrix} s11(x1,x0) & 0 & 0 & s14(x1,x0) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ s14(x1,x0) & 0 & 0 & s44(x1,x0) \end{bmatrix}$$

"Boomerang" torsion: another special case.

```
> eq2:=eval(All({
  phi12(x[-j])=chi12*(x1-x2),phi13(x[-j])=chi13*(x1-x3),
  phi23(x[-j])=chi23*(x2-x3),phi14(x[-j])=chi14*(x0-x1),
  phi24(x[-j])=chi24*(x0-x2),phi34(x[-j])=chi34*(x0-x3),
  s11(x[-j])=psi11*(x1+x1),s12(x[-j])=psi12*(x1+x2),
  s13(x[-j])=psi13*(x1+x3),s14(x[-j])=psi14*(x1+x0),
  s22(x[-j])=psi22*(x2+x2),s23(x[-j])=psi23*(x2+x3),
  s24(x[-j])=psi24*(x2+x0),s33(x[-j])=psi33*(x3+x3),
```

```
s34(x[-j])=psi34*(x3+x0),s44(x[-j])=psi44*(x0+x0)
})):
> eq3:={chi12=chi12,chi13=0,chi14=0,
chi23=0,chi24=0,
chi34=0,
psi11=psi11,psi12=0,psi13=0,psi14=0,
psi22=1,psi23=0,psi24=0,
psi33=1,psi34=0,
psi44=1}:
> let('s2[-a,i]=subs(
eq1,
#eq2,
#eq3,
s[-a,i])',simplify);
```

$$s2_{-a,\,i} = \mathrm{subs}(eq1,\, s_{-a,\,i})$$

```
> let('q2[-a,b]=subs(
eq1,
#eq2,
#eq3,
q[-a,b])',simplify);
```

$$q2_{-a,\,b} = \mathrm{subs}(eq1,\, q_{-a,\,b})$$

General tetrad

```
> infolevel[let]:=0:
let(e[-a,i]=q2[-a,b]*s2[-b,i],simplify);
```

$$e_{-a,\,i} = q2_{-a,\,b}\, s2_{-b,\,i}$$

Compute the inverse frame matrix (covariant tetrad)

Note: Avoid re-evaluation of inverse by doing it first.

```
> infolevel[IS]:=0:
inverse(matrix(dimension$2,(b,j)->e[-b,j])):
let(E[-i,a]=%[i,a],simplify):
```

Check orthogonality conditions

```
> `eq5.1a`:=IS(E[-i,a]*e[-a,j]=piecewise(i=j,1,0),simplify);
```

$$true$$

$$eq5.1a := E_{-i,\,a}\, e_{-a,\,j} = \begin{cases} 1 & i = j \\ 0 & otherwise \end{cases}$$

```
> infolevel[IS]:=0:
  `eq5.1b`:=IS(E[-i,a]*e[-b,i]=piecewise(a=b,1,0),simplify);
```

$$true$$

$$eq5.1b := E_{-i,\,a}\, e_{-b,\,i} = \begin{cases} 1 & a = b \\ 0 & otherwise \end{cases}$$

Minkowski metric (pseudo-euclidean space)

```
> let('eta[a,b]=[
  [-1,0,0,0],
  [0,-1,0,0],
  [0,0,-1,0],
  [0,0,0,1]
  ][a,b]'):
```

Riemann metric

```
> `eq5.4`:=let('G[i,k]=e[-a,i]*e[-b,k]*eta[a,b]',simplify);
```

$$eq5.4 := G_{i,\,k} = e_{-a,\,i}\, e_{-b,\,k}\, \eta_{a,\,b}$$

```
> matrix(dimension$2,(i,j)->G[i,j]);
```

$$\begin{bmatrix} \mathrm{s14}(x1,x0)^2 - \mathrm{s11}(x1,x0)^2, & 0, & 0, & \mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0) - \mathrm{s14}(x1,x0)\,\mathrm{s11}(x1,x0) \\ 0, & -1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ \mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0) - \mathrm{s14}(x1,x0)\,\mathrm{s11}(x1,x0), & 0, & 0, & \mathrm{s44}(x1,x0)^2 - \mathrm{s14}(x1,x0)^2 \end{bmatrix}$$

```
> `eq5.4`:=IS('G[i,k]=s2[-a,i]*s2[-b,k]*eta[a,b]',simplify);
```

$$true$$

$$eq5.4 := G_{i,\,k} = s2_{-a,\,i}\, s2_{-b,\,k}\, \eta_{a,\,b}$$

Inverse Riemann metric

```
> inverse(matrix(dimension$2,(i,j)->G[i,j])):
  let(g[-i,-j]=%[i,j],simplify):
> matrix(dimension$2,(i,j)->(g[-i,-j]));
```

$$\left[\frac{-\mathrm{s44}(x1,x0)^2 + \mathrm{s14}(x1,x0)^2}{\mathrm{s14}(x1,x0)^4 + \mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2 - 2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)}, 0, 0, -\frac{\mathrm{s14}(x1,x0)\,(-\mathrm{s44}(x1,x0) + \mathrm{s11}(x1,x0))}{\mathrm{s14}(x1,x0)^4 + \mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2 - 2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)} \right]$$

$$[0, -1, 0, 0]$$

$$[0, 0, -1, 0]$$

$$\left[-\frac{\mathrm{s14}(x1,x0)\,(-\mathrm{s44}(x1,x0)+\mathrm{s11}(x1,x0))}{\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)}, 0, 0, \frac{-\mathrm{s14}(x1,x0)^2+\mathrm{s11}(x1,x0)^2}{\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)}\right]$$

```
> eq3:=IS('G[i,k]*g[-k,-l]=piecewise(i=l,1,0)',simplify);
```

$$\mathit{true}$$

$$eq3 := G_{i,\,k}\, g_{-k,\,-l} = \begin{cases} 1 & i = l \\ 0 & \mathit{otherwise} \end{cases}$$

Christoffel symbol

```
> `(5.6)`:='Gamma[-i,j,k]=1/2*g[-i,-m]*
  (der(x[-k])(G[j,m])+der(x[-j])(G[k,m])-der(x[-m])(G[j,k]))';
```

$$(5.6) := \Gamma_{-i,\,j,\,k} = \frac{1}{2}\, g_{-i,\,-m}\,(\mathrm{D}_k(G_{j,\,m}) + \mathrm{D}_j(G_{k,\,m}) - \mathrm{D}_m(G_{j,\,k}))$$

```
> let(`(5.6)`,simplify);
```

$$\Gamma_{-i,\,j,\,k} = \frac{1}{2}\, g_{-i,\,-m}\left(\left(\frac{\partial}{\partial x_{-k}} G_{j,\,m}\right) + \left(\frac{\partial}{\partial x_{-j}} G_{k,\,m}\right) - \left(\frac{\partial}{\partial x_{-m}} G_{j,\,k}\right)\right)$$

In orthonormal frames the Christoffel symbols vanish. But symmetric terms make them non-zero.

```
> 'matrix(dimension$2,(j,k)->(Gamma[-i,j,k]))'$i=1..dimension;
```

$$\Bigg[\Bigg(\mathrm{s44}(x1,x0)^2\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s11}(x1,x0)\right) + \mathrm{s14}(x1,x0)^3\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)$$
$$+\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\mathrm{s14}(x1,x0)^2$$
$$-\,2\,\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\mathrm{s11}(x1,x0)$$
$$-\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s11}(x1,x0)\right) - \mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)$$
$$+\,\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$-\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\mathrm{s14}(x1,x0)^2 + \mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\mathrm{s11}(x1,x0)^2$$

$$+ \mathrm{s11}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) - \mathrm{s14}(x1, x0)\, \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)\Bigg)$$

$$\Big/ \left(\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\right), 0, 0$$

$$, \Bigg(-\mathrm{s44}(x1, x0)^2\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \mathrm{s44}(x1, x0)^2\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) - \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)\, \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) - \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)\, \mathrm{s11}(x1, x0)\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)\Bigg) \Big/ \Big($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\Big)\Bigg]$$

$$[0, 0, 0, 0]$$

$$[0, 0, 0, 0]$$

$$\Bigg[\Bigg(-\mathrm{s44}(x1, x0)^2\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \mathrm{s44}(x1, x0)^2\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) - \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)\, \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) - \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)\, \mathrm{s11}(x1, x0)\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)\Bigg) \Big/ \Big($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\Big), 0, 0, -\Bigg($$

$$\mathrm{s44}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) - \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) \mathrm{s11}(x1, x0)$$

$$- \mathrm{s44}(x1, x0)^2\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) - \mathrm{s44}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s44}(x1, x0)^2\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) - \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) + \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0)^2$$

$$- \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) + \mathrm{s14}(x1, x0)\, \mathrm{s11}(x1, x0)\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s44}(x1, x0)\right)\Bigg)$$

$$\Big/ \left(\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\right)\Bigg],$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Bigg[-\Bigg(\mathrm{s14}(x1, x0)\, \mathrm{s44}(x1, x0)\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s11}(x1, x0)\right) + \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s11}(x1, x0)\right) - \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) - \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0)$$

$$- \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) + \mathrm{s11}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^2\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) - \mathrm{s11}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)\Bigg)\Bigg/ ($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)), 0, 0, \Bigg($$

$$\mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)\, \mathrm{s44}(x1, x0)\, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$- \mathrm{s11}(x1, x0)\, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \mathrm{s14}(x1, x0)\, \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$- \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0)^2 + \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) - \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) \mathrm{s11}(x1, x0)^2\Bigg)$$

$$\Big/ \left(\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\right)\Bigg]$$

$[0, 0, 0, 0]$

$[0, 0, 0, 0]$

$$\left[\left(\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\right.\right.$$
$$-\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$-\mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)+\mathrm{s14}(x1,x0)\,\mathrm{s11}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$-\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\mathrm{s14}(x1,x0)^2+\mathrm{s14}(x1,x0)^3\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)$$
$$\left.+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)-\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\mathrm{s11}(x1,x0)^2\right)$$
$$\Big/\left(\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)\right),0,0$$
$$,-\left(-\mathrm{s44}(x1,x0)^2\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\right.$$
$$+2\,\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\mathrm{s11}(x1,x0)$$
$$+\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)+\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)$$
$$-\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)+\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s44}(x1,x0)\right)\mathrm{s14}(x1,x0)^2$$
$$-\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$-\mathrm{s14}(x1,x0)\,\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)$$
$$+\mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)-\mathrm{s14}(x1,x0)^3\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)$$
$$\left.\left.-\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s44}(x1,x0)\right)\right)\right)\Big/\Big($$
$$\left.\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)\right)\Big)\right]$$

Affine Connection

```
> `(5.24)`:='Gamma0[-i,j,k]=E[-i,a]*der(x[-k])(e[-a,j])';
```

$$(5.24) := \Gamma 0_{-i,\,j,\,k} = E_{-i,\,a}\,\mathrm{D}_k(e_{-a,\,j})$$

```
> let(`(5.24)`,simplify);
```

$$\Gamma 0_{-i,\,j,\,k} = E_{-i,\,a}\left(\frac{\partial}{\partial x_{-k}} e_{-a,\,j}\right)$$

Shipov Proposition 5.1: Parallel displacement (covariant derivative) of the tetrad equals zero.

```
> `(5.22)`:='der(x[-k],Gamma0)(E[-i,a])=0';
```

$$(5.22) := (\mathrm{D}_k^{\Gamma 0})(E_{-i,\,a}) = 0$$

```
> IS(`(5.22)`,simplify);
```

$$true$$

$$\left(\frac{\partial}{\partial x_{-k}} E_{-i,\,a}\right) + \Gamma 0_{-i,\;j',\,k}\, E_{-j',\,a} = 0$$

```
> `(5.23)`:='der(x[-k],Gamma0)(e[-a,j])=0';
```

$$(5.23) := (\mathrm{D}_k^{\Gamma 0})(e_{-a,\,j}) = 0$$

```
> IS(`(5.23)`,simplify);
```

$$true$$

$$\left(\frac{\partial}{\partial x_{-k}} e_{-a,\,j}\right) - \Gamma 0_{-j',\,j,\,k}\, e_{-a,\;j'} = 0$$

```
> 'matrix(dimension$2,(j,k)->Gamma0[-i,j,k])'$i=1..dimension;
```

$$\Bigg[\Bigg(\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s11}(x1,x0) - \mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)$$
$$+\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s11}(x1,x0)\right) - \mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s14}(x1,x0)\Bigg) \Big/ \Big($$
$$-\mathrm{s14}(x1,x0)^2 + \mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)\Big), 0, 0, \Bigg(\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)\mathrm{s11}(x1,x0)$$
$$-\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right) + \mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$-\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)\mathrm{s14}(x1,x0)\Bigg) \Big/ \Big(-\mathrm{s14}(x1,x0)^2 + \mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)\Big)\Bigg]$$

$$[0, 0, 0, 0]$$

$$[0, 0, 0, 0]$$

$$\Bigg[\Bigg(\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right) - \mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)$$
$$+\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right) - \mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\Bigg) \Big/ \Big($$

$$-\mathrm{s14}(x1,x0)^2+\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)),0,0,-\left(-\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\right.$$
$$+\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)-\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)$$
$$\left.+\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s44}(x1,x0)\right)\right)\Big/(-\mathrm{s14}(x1,x0)^2+\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0))\Big],$$
$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},$$
$$\Big[\left(\mathrm{s11}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)-\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\right.$$
$$\left.+\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s11}(x1,x0)\right)-\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\right)\Big/($$
$$\mathrm{s14}(x1,x0)^2-\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)),0,0,-\left(-\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)\right.$$
$$+\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)-\mathrm{s11}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)$$
$$\left.+\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\right)\Big/(\mathrm{s14}(x1,x0)^2-\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0))\Big]$$
$$[0,0,0,0]$$
$$[0,0,0,0]$$
$$\Big[\left(\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s11}(x1,x0)-\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\right.$$
$$\left.+\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)-\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s14}(x1,x0)\right)\Big/($$
$$\mathrm{s14}(x1,x0)^2-\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)),0,0,\left(-\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)\mathrm{s11}(x1,x0)\right.$$
$$+\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s44}(x1,x0)\right)-\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)$$
$$\left.+\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\phi 14(x1,x0)\right)\mathrm{s14}(x1,x0)\right)\Big/(-\mathrm{s14}(x1,x0)^2+\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0))\Big]$$

Riemann-Christoffel curvature tensor

```
> `(5.49)`:='S[-i,j,k,m]=2*alt[m,k](der(x[-k])(Gamma0[-i,j,m]))+2
  *alt[k,m](Gamma0[-i,s,k]*Gamma0[-s,j,m])';
```

$$(5.49) := S_{-i,\,j,\,k,\,m} = 2\,alt_{m,\,k}(\mathrm{D}_k(\Gamma 0_{-i,\,j,\,m})) + 2\,alt_{k,\,m}(\Gamma 0_{-i,\,s,\,k}\,\Gamma 0_{-s,\,j,\,m})$$

```
> let(`(5.49)`,simplify);
```

$$S_{-i,\,j,\,k,\,m} = \left(\frac{\partial}{\partial x_{-k}}\Gamma 0_{-i,\,j,\,m}\right) - \left(\frac{\partial}{\partial x_{-m}}\Gamma 0_{-i,\,j,\,k}\right) + \Gamma 0_{-i,\,s,\,k}\,\Gamma 0_{-s,\,j,\,m} - \Gamma 0_{-i,\,s,\,m}\,\Gamma 0_{-s,\,j,\,k}$$

Proposition 5.4, page 192. The Riemann-Christoffel tensor equals zero.

```
> `(5.52)`:=IS(S[-i,j,k,m]=0,simplify);
```

$$true$$

$$(5.52) := S_{-i,\,j,\,k,\,m} = 0$$

Proposition 5.3 The Ricci rotation coefficients

```
> `(5.38)`:='Omega0[-i,j,k]=E[-i,a]*der(x[-k],Gamma)(e[-a,j])';
```

$$(5.38) := \Omega 0_{-i,\,j,\,k} = E_{-i,\,a}\,(\mathrm{D}_k^{\Gamma})(e_{-a,\,j})$$

```
> let(`(5.38)`,simplify,x->normal(x,expanded));
```

$$\Omega 0_{-i,\,j,\,k} = E_{-i,\,a}\left(\left(\frac{\partial}{\partial x_{-k}} e_{-a,\,j}\right) - \Gamma_{-j',\,j,\,k}\,e_{-a,\,j'}\right)$$

```
> 'matrix(dimension$2,(j,k)->
  Omega0[-i,j,k]
  )'$i=1..dimension;
```

$$\Big[-\mathrm{s14}(x1,x0)\Big(-\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)$$
$$+\mathrm{s11}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right) - \mathrm{s11}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$
$$+\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s44}(x1,x0)^2 - \mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\mathrm{s44}(x1,x0)$$
$$+\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)\mathrm{s44}(x1,x0) - \mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)$$
$$+\mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right) + \mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)$$
$$+\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\mathrm{s44}(x1,x0) - \mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\mathrm{s44}(x1,x0)$$
$$-\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi 14(x1,x0)\right)\mathrm{s44}(x1,x0)\Big)\Big/\,($$

$$\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0))\,,0\,,0\,,$$

$$\mathrm{s14}(x1,x0)\left(-\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi14(x1,x0)\right)\mathrm{s11}(x1,x0)+\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\right.$$

$$+\left(\frac{\partial}{\partial x0}\phi14(x1,x0)\right)\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)-\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x0}\phi14(x1,x0)\right)\mathrm{s11}(x1,x0)$$

$$+\mathrm{s11}(x1,x0)\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)+\mathrm{s14}(x1,x0)\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$

$$-\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)+\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)$$

$$-\mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)-\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$

$$\left.-\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)\mathrm{s44}(x1,x0)\,\mathrm{s11}(x1,x0)+\mathrm{s14}(x1,x0)^2\,\mathrm{s44}(x1,x0)\left(\frac{\partial}{\partial x0}\phi14(x1,x0)\right)\right)$$

$$\left.\Big/\,(\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0))\right]$$

$$[0\,,0\,,0\,,0]$$

$$[0\,,0\,,0\,,0]$$

$$\left[-\left(-\mathrm{s14}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi14(x1,x0)\right)\right.\right.$$

$$-\mathrm{s44}(x1,x0)^2\,\mathrm{s14}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)+\mathrm{s14}(x1,x0)^4\left(\frac{\partial}{\partial x1}\phi14(x1,x0)\right)$$

$$+\mathrm{s44}(x1,x0)^3\left(\frac{\partial}{\partial x1}\phi14(x1,x0)\right)\mathrm{s11}(x1,x0)+\mathrm{s44}(x1,x0)^2\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)$$

$$+\mathrm{s14}(x1,x0)^3\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)-\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\phi14(x1,x0)\right)\mathrm{s44}(x1,x0)\,\mathrm{s11}(x1,x0)$$

$$-\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0)\left(\frac{\partial}{\partial x0}\mathrm{s11}(x1,x0)\right)+\mathrm{s14}(x1,x0)\,\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)$$

$$+\mathrm{s11}(x1,x0)\,\mathrm{s14}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)-\mathrm{s44}(x1,x0)^2\left(\frac{\partial}{\partial x1}\mathrm{s14}(x1,x0)\right)\mathrm{s11}(x1,x0)$$

$$\left.\left.-\mathrm{s14}(x1,x0)^3\left(\frac{\partial}{\partial x1}\mathrm{s44}(x1,x0)\right)\right)\right.\Big/\,($$

$$\mathrm{s14}(x1,x0)^4+\mathrm{s11}(x1,x0)^2\,\mathrm{s44}(x1,x0)^2-2\,\mathrm{s44}(x1,x0)\,\mathrm{s14}(x1,x0)^2\,\mathrm{s11}(x1,x0))\,,0\,,0\,,-\Big($$

$$\mathrm{s44}(x1,x0)^3\left(\frac{\partial}{\partial x0}\phi14(x1,x0)\right)\mathrm{s11}(x1,x0)-\mathrm{s44}(x1,x0)^3\left(\frac{\partial}{\partial x0}\mathrm{s14}(x1,x0)\right)$$

$$+ \mathrm{s44}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) - \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s44}(x1, x0)^2 \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) - \mathrm{s44}(x1, x0)^2 \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) - \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0)^2$$

$$+ \mathrm{s44}(x1, x0) \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \mathrm{s14}(x1, x0)^4 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0) \mathrm{s11}(x1, x0)$$

$$- \mathrm{s14}(x1, x0)^2 \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right)\Big)\Big/ ($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2 \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0) \mathrm{s14}(x1, x0)^2 \mathrm{s11}(x1, x0))\Big],$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Big[-\Big(\mathrm{s14}(x1, x0)^4 \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) + \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0) \mathrm{s11}(x1, x0)$$

$$- \mathrm{s14}(x1, x0)^2 \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) + \mathrm{s11}(x1, x0) \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) \mathrm{s14}(x1, x0)^2 - \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$- \mathrm{s11}(x1, x0)^2 \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \mathrm{s11}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0) - \mathrm{s11}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^3 \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right)\Big)\Big/ ($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2 \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0) \mathrm{s14}(x1, x0)^2 \mathrm{s11}(x1, x0)), 0, 0, -\Big($$

$$-\mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) \mathrm{s44}(x1, x0) - \mathrm{s14}(x1, x0)^2 \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0) - \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0)^2 \, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) + \mathrm{s14}(x1, x0)^3 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s14}(x1, x0)^2 + \mathrm{s44}(x1, x0) \, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) \mathrm{s11}(x1, x0)^2 + \mathrm{s14}(x1, x0) \, \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0)^4 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) - \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0) \, \mathrm{s11}(x1, x0)$$

$$\Big) \Big/ \left(\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2 \, \mathrm{s44}(x1, x0)^2 - 2 \, \mathrm{s44}(x1, x0) \, \mathrm{s14}(x1, x0)^2 \, \mathrm{s11}(x1, x0)\right) \Big]$$

$[0, 0, 0, 0]$

$[0, 0, 0, 0]$

$$\Big[\mathrm{s14}(x1, x0) \Big(-\mathrm{s11}(x1, x0)^2 \, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) + \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s11}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) + \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0)^2$$

$$- \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) \mathrm{s44}(x1, x0) + \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) \mathrm{s44}(x1, x0)$$

$$- \mathrm{s11}(x1, x0) \, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) + \mathrm{s11}(x1, x0) \, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right)$$

$$+ \mathrm{s11}(x1, x0) \, \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) + \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right) \mathrm{s44}(x1, x0)$$

$$- \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) \mathrm{s44}(x1, x0) - \mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \phi 14(x1, x0)\right) \mathrm{s44}(x1, x0) \Big)$$

$$\Big/ \left(\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2 \, \mathrm{s44}(x1, x0)^2 - 2 \, \mathrm{s44}(x1, x0) \, \mathrm{s14}(x1, x0)^2 \, \mathrm{s11}(x1, x0)\right), 0, 0$$

$$, - \mathrm{s14}(x1, x0) \Big(-\mathrm{s14}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s11}(x1, x0)$$

$$+ \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) + \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s11}(x1, x0)^2 \, \mathrm{s44}(x1, x0)$$

$$- \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right) \mathrm{s11}(x1, x0) + \mathrm{s11}(x1, x0) \, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s14}(x1, x0) \, \mathrm{s11}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) - \mathrm{s44}(x1, x0)^2 \left(\frac{\partial}{\partial x1} \mathrm{s44}(x1, x0)\right)$$

$$+ \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right) - \mathrm{s11}(x1, x0)\, \mathrm{s14}(x1, x0) \left(\frac{\partial}{\partial x1} \mathrm{s14}(x1, x0)\right)$$

$$- \mathrm{s14}(x1, x0)\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x0} \mathrm{s11}(x1, x0)\right) - \left(\frac{\partial}{\partial x0} \mathrm{s14}(x1, x0)\right) \mathrm{s44}(x1, x0)\, \mathrm{s11}(x1, x0)$$

$$+ \mathrm{s14}(x1, x0)^2\, \mathrm{s44}(x1, x0) \left(\frac{\partial}{\partial x0} \phi 14(x1, x0)\right)\Bigg) \Big/ \Big($$

$$\mathrm{s14}(x1, x0)^4 + \mathrm{s11}(x1, x0)^2\, \mathrm{s44}(x1, x0)^2 - 2\, \mathrm{s44}(x1, x0)\, \mathrm{s14}(x1, x0)^2\, \mathrm{s11}(x1, x0)\Big)\Big]$$

Proposition 5.2 Connection $\Gamma 0_{-i,\, j,\, k}$ can be represented as the sum

```
> `(5.28)`:=IS(Gamma0[-i,j,k]=Gamma[-i,j,k]+Omega0[-i,j,k],
  simplify);
```

$$true$$

$$(5.28) := \Gamma 0_{-i,\, j,\, k} = \Gamma_{-i,\, j,\, k} + \Omega 0_{-i,\, j,\, k}$$

Riemann tensor (curvature of V_4)

```
> `(5.54a)`:='R[-i,j,k,m]=2*alt[m,k](der(x[-k])(Gamma[-i,j,m]))+2
  *alt[k,m](Gamma[-i,s,k]*Gamma[-s,j,m])';
```

$$(5.54a) := R_{-i,\, j,\, k,\, m} = 2\, alt_{m,\, k}(\mathrm{D}_k(\Gamma_{-i,\, j,\, m})) + 2\, alt_{k,\, m}(\Gamma_{-i,\, s,\, k}\, \Gamma_{-s,\, j,\, m})$$

```
> let(`(5.54a)`,simplify);
```

$$R_{-i,\, j,\, k,\, m} = \left(\frac{\partial}{\partial x_{-k}} \Gamma_{-i,\, j,\, m}\right) - \left(\frac{\partial}{\partial x_{-m}} \Gamma_{-i,\, j,\, k}\right) + \Gamma_{-i,\, s,\, k}\, \Gamma_{-s,\, j,\, m} - \Gamma_{-i,\, s,\, m}\, \Gamma_{-s,\, j,\, k}$$

Ricci curvature (compare to Bergmann spin curvature)

```
> `(5.54b)`:='R0[-i,j,k,m]=2*alt[m,k](der(x[-k],Gamma)(Omega0[-i,
  j,m]))+2*alt[k,m](Omega0[-i,s,k]*Omega0[-s,j,m])';
```

$$(5.54b) := R0_{-i,\, j,\, k,\, m} = 2\, alt_{m,\, k}((\mathrm{D}_k^{\Gamma})(\Omega 0_{-i,\, j,\, m})) + 2\, alt_{k,\, m}(\Omega 0_{-i,\, s,\, k}\, \Omega 0_{-s,\, j,\, m})$$

```
> let(`(5.54b)`,simplify);
```

$$R0_{-i,\, j,\, k,\, m} = \left(\frac{\partial}{\partial x_{-k}} \Omega 0_{-i,\, j,\, m}\right) - \Gamma_{-j',\, j,\, k}\, \Omega 0_{-i,\, j',\, m} - \Gamma_{-j',\, m,\, k}\, \Omega 0_{-i,\, j,\, j'} + \Gamma_{-i,\, j',\, k}\, \Omega 0_{-j',\, j,\, m}$$

$$- \left(\frac{\partial}{\partial x_{-m}} \Omega 0_{-i,\, j,\, k}\right) + \Gamma_{-j',\, j,\, m}\, \Omega 0_{-i,\, j',\, k} + \Gamma_{-j',\, k,\, m}\, \Omega 0_{-i,\, j,\, j'} - \Gamma_{-i,\, j',\, m}\, \Omega 0_{-j',\, j,\, k} + \Omega 0_{-i,\, s,\, k}\, \Omega 0_{-s,\, j,\, m}$$

$$- \Omega 0_{-i,\, s,\, m}\, \Omega 0_{-s,\, j,\, k}$$

Proposition 5.5 The Cartan curvature tensor $S_{-i,\,j,\,k,\,m}$ can be represented as the sum of the Riemann curvature and the Ricci curvature.

```
> `(5.53)`:=IS(S[-i,j,k,m]=R[-i,j,k,m]+R0[-i,j,k,m],
  simplify);
```

$$true$$

$$(5.53) := S_{-i,\,j,\,k,\,m} = R_{-i,\,j,\,k,\,m} + R0_{-i,\,j,\,k,\,m}$$

Note that there are errors in Shipov's eqs. (5.56, 5.58 and 5.59).

```
> `(5.59)`:='R[-i,j,k,m]=2*E[-i,a]*alt[m,k](der(x[-m],Gamma)(der(
  x[-k],Gamma)(e[-a,j])))';
```

$$(5.59) := R_{-i,\,j,\,k,\,m} = 2\,E_{-i,\,a}\,alt_{m,\,k}((\mathrm{D}_m{}^{\Gamma})((\mathrm{D}_k{}^{\Gamma})(e_{-a,\,j})))$$

```
> IS(`(5.59)`,simplify);
```

$$false$$

$$\begin{aligned}
R_{-i,\,j,\,k,\,m} = {} & 2\,E_{-i,\,a}\left(\frac{1}{2}\left(\frac{\partial}{\partial x_{-m}}\left(\left(\frac{\partial}{\partial x_{-k}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,k}\,e_{-a,\,j'}\right)\right)\right. \\
& -\frac{1}{2}\Gamma_{-j',\,j',\,m}\left(\left(\frac{\partial}{\partial x_{-k}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,k}\,e_{-a,\,j'}\right)-\frac{1}{2}\Gamma_{-j',\,k,\,m}\left(\left(\frac{\partial}{\partial x_{-j'}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,j'}\,e_{-a,\,j'}\right) \\
& -\frac{1}{2}\Gamma_{-j',\,j,\,m}\left(\left(\frac{\partial}{\partial x_{-k}}e_{-a,\,j'}\right)-\Gamma_{-j',\,j',\,k}\,e_{-a,\,j'}\right)+\frac{1}{2}\Gamma_{-j',\,j',\,m}\left(\left(\frac{\partial}{\partial x_{-k}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,k}\,e_{-a,\,j'}\right) \\
& -\frac{1}{2}\left(\frac{\partial}{\partial x_{-k}}\left(\left(\frac{\partial}{\partial x_{-m}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,m}\,e_{-a,\,j'}\right)\right)+\frac{1}{2}\Gamma_{-j',\,j',\,k}\left(\left(\frac{\partial}{\partial x_{-m}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,m}\,e_{-a,\,j'}\right) \\
& +\frac{1}{2}\Gamma_{-j',\,m,\,k}\left(\left(\frac{\partial}{\partial x_{-j'}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,j'}\,e_{-a,\,j'}\right)+\frac{1}{2}\Gamma_{-j',\,j,\,k}\left(\left(\frac{\partial}{\partial x_{-m}}e_{-a,\,j'}\right)-\Gamma_{-j',\,j',\,m}\,e_{-a,\,j'}\right) \\
& \left.-\frac{1}{2}\Gamma_{-j',\,j',\,k}\left(\left(\frac{\partial}{\partial x_{-m}}e_{-a,\,j}\right)-\Gamma_{-j',\,j,\,m}\,e_{-a,\,j'}\right)\right)
\end{aligned}$$

Consider a general contravariant vector field

```
> let(f[-i]=[f1(all(x[-j])),f2(all(x[-j])),f3(all(x[-j])),f0(all(
  x[-j]))][i]);
```

$$f_{-i} = [\,\mathrm{f1}(x1, x2, x3, x0), \mathrm{f2}(x1, x2, x3, x0), \mathrm{f3}(x1, x2, x3, x0), \mathrm{f0}(x1, x2, x3, x0)\,]_i$$

Its convariant derivative is

```
> 'df[-i,k]=der(x[-k],Gamma0)(f[-i])';
```

$$df_{-i,\,k} = (\mathrm{D}_k^{\Gamma 0})(f_{-i})$$

```
> let(%,simplify);
```

$$df_{-i,\,k} = \left(\frac{\partial}{\partial x_{-k}} f_{-i}\right) + \Gamma 0_{-i,\;j',\,k}\, f_{-j'}$$

```
> #matrix(dimension$2,(i,k)->df[-i,k]);
```

We will compare this convariant derivative with that calculated using Bergmann's methods below.

Define the 4-d Levi-Civita symbol

```
> delta:=array(tensor,antisymmetric,1..4,1..4,1..4,1..4);
```

$$\delta := \mathrm{array}(\mathit{tensor}, \mathit{antisymmetric}, 1\,..\,4, 1\,..\,4, 1\,..\,4, 1\,..\,4, [\ \ ])$$

```
> delta[1,2,3,4]:=1;
```

$$\delta_{1,\,2,\,3,\,4} := 1$$

Define the contravariant components of the spin matrices based on the frame matrix given above.

```
> u:=array(-4..4):
  let(u[-j]=E[-j,4]):
> alpha:=array(-4..4):
  let(alpha[-j]=E[-j,1]):
> beta:=array(-4..4):
  let(beta[-j]=E[-j,2]):
> gamma:=array(-4..4):
  let(gamma[-j]=E[-j,3]):
> #vector(4,i->u[-i]);
  #vector(4,i->alpha[-i]);
  #vector(4,i->beta[-i]);
  #vector(4,i->gamma[-i]);
```

Compute the determinant
Ref. Bergmann eq. (2.16)

```
> `(2.16a)`:=let('Delta=delta[mu,iota,kappa,lambda]*u[-mu]*alpha[
  -iota]*beta[-kappa]*gamma[-lambda]',simplify);
```

$$(2.16a) := \Delta = \delta_{\mu,\,\iota,\,\kappa,\,\lambda}\, u_{-\mu}\, \alpha_{-\iota}\, \beta_{-\kappa}\, \gamma_{-\lambda}$$

```
> Delta;
```

$$\frac{1}{\mathrm{s14}(x1, x0)^2 - \mathrm{s11}(x1, x0)\,\mathrm{s44}(x1, x0)}$$

Compute the covariant components

```
> `(2.16b)`:=let('u[mu]=1/Delta*delta[mu,iota,kappa,lambda]*alpha
  [-iota]*beta[-kappa]*gamma[-lambda]',simplify);
```

$$(2.16b) := u_{\mu} = \frac{\delta_{\mu,\,\iota,\,\kappa,\,\lambda}\,\alpha_{-\iota}\,\beta_{-\kappa}\,\gamma_{-\lambda}}{\Delta}$$

```
> `(2.16c)`:=let('alpha[mu]=1/Delta*delta[mu,iota,kappa,lambda]*u
  [-iota]*beta[-kappa]*gamma[-lambda]',simplify);
```

$$(2.16c) := \alpha_{\mu} = \frac{\delta_{\mu,\,\iota,\,\kappa,\,\lambda}\,u_{-\iota}\,\beta_{-\kappa}\,\gamma_{-\lambda}}{\Delta}$$

```
> `(2.16d)`:=let('beta[mu]=1/Delta*delta[mu,iota,kappa,lambda]*u[
  -kappa]*alpha[-iota]*gamma[-lambda]',simplify);
```

$$(2.16d) := \beta_{\mu} = \frac{\delta_{\mu,\,\iota,\,\kappa,\,\lambda}\,u_{-\kappa}\,\alpha_{-\iota}\,\gamma_{-\lambda}}{\Delta}$$

```
> `(2.16e)`:=let('gamma[mu]=1/Delta*delta[mu,iota,kappa,lambda]*u
  [-lambda]*alpha[-iota]*beta[-kappa]',simplify);
```

$$(2.16e) := \gamma_{\mu} = \frac{\delta_{\mu,\,\iota,\,\kappa,\,\lambda}\,u_{-\lambda}\,\alpha_{-\iota}\,\beta_{-\kappa}}{\Delta}$$

Compute the contravariant and covariant metric. Check that it is the same as the conventional expression.

```
> `(2.15)`:=IS(g[-mu,-nu]=u[-mu]*u[-nu]-(alpha[-mu]*alpha[-nu]+be
  ta[-mu]*beta[-nu]+gamma[-mu]*gamma[-nu]),simplify);
```

$$true$$

$$(2.15) := g_{-\mu,\,-\nu} = u_{-\mu}\,u_{-\nu} - \alpha_{-\mu}\,\alpha_{-\nu} - \beta_{-\mu}\,\beta_{-\nu} - \gamma_{-\mu}\,\gamma_{-\nu}$$

```
> `(2.17)`:=IS(G[mu,nu]=u[mu]*u[nu]-(alpha[mu]*alpha[nu]+beta[mu]
  *beta[nu]+gamma[mu]*gamma[nu]),simplify);
```

$$true$$

$$(2.17) := G_{\mu,\,\nu} = u_{\mu}\,u_{\nu} - \alpha_{\mu}\,\alpha_{\nu} - \beta_{\mu}\,\beta_{\nu} - \gamma_{\mu}\,\gamma_{\nu}$$

Bergmann's Hermitean matrix basis * 2^(-1/2).

```
> Sigma:=array(-4..4):
> let(Sigma[-a]=2^(-1/2)*[
  matrix([[1,0],[0,-1]]),
  matrix([[0,1],[1,0]]),
```

```
matrix([[0,-I],[I,0]]),
matrix([[1,0],[0,1]])
]
[a]);
```

$$\Sigma_{-a} = \frac{1}{2}\sqrt{2}\left[\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & -I\\ I & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right]_a$$

```
> let(Sigma[a]=2^(-1/2)*[
matrix([[-1,0],[0,1]]),
matrix([[0,-1],[-1,0]]),
matrix([[0,I],[-I,0]]),
matrix([[1,0],[0,1]])
]
[a]);
```

$$\Sigma_{a} = \frac{1}{2}\sqrt{2}\left[\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}, \begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right]_a$$

2) For efficiency, instead of using the TensorLite tensor array data type, we pre-calculate both the covariant and contravariant components and store them in an array with both contravariant and covariant bounds.

```
> sigma:=array(-4..4):
```

Spin frame

```
> `(2.13a)`:=let('sigma[-mu]=matrix([
[u[-mu]+alpha[-mu], beta[-mu]-I*gamma[-mu]],
[beta[-mu]+I*gamma[-mu], u[-mu]-alpha[-mu]]
])');
```

$$(2.13a) := \sigma_{-\mu} = \begin{bmatrix} u_{-\mu} + \alpha_{-\mu} & \beta_{-\mu} - I\,\gamma_{-\mu} \\ \beta_{-\mu} + I\,\gamma_{-\mu} & u_{-\mu} - \alpha_{-\mu} \end{bmatrix}$$

```
> IS(2^(1/2)*Sigma[-a]*E[-mu,a]=sigma[-mu],simplify);
```

$$true$$

$$\sqrt{2}\,\Sigma_{-a}\,E_{-\mu,\,a} = \sigma_{-\mu}$$

```
> `(2.13b)`:=let('sigma[mu]=matrix([
[u[mu]+alpha[mu], beta[mu]-I*gamma[mu]],
[beta[mu]+I*gamma[mu], u[mu]-alpha[mu]]
])');
```

$$(2.13b) := \sigma_{\mu} = \begin{bmatrix} u_{\mu} + \alpha_{\mu} & \beta_{\mu} - I\,\gamma_{\mu} \\ \beta_{\mu} + I\,\gamma_{\mu} & u_{\mu} - \alpha_{\mu} \end{bmatrix}$$

```
> IS('2^(1/2)*Sigma[a]*e[-a,mu]=sigma[mu]',simplify);
```

$$true$$

$$\sqrt{2}\,\Sigma_a\, e_{-a,\,\mu} = \sigma_\mu$$

```
> tau:=array(-4..4):
  `(2.14a)`:=let('tau[-mu]=matrix([
  [-u[-mu]+alpha[-mu], beta[-mu]-I*gamma[-mu]],
  [beta[-mu]+I*gamma[-mu], -u[-mu]-alpha[-mu]]
  ])');
```

$$(2.14a) := \tau_{-\mu} = \begin{bmatrix} -u_{-\mu} + \alpha_{-\mu} & \beta_{-\mu} - I\,\gamma_{-\mu} \\ \beta_{-\mu} + I\,\gamma_{-\mu} & -u_{-\mu} - \alpha_{-\mu} \end{bmatrix}$$

```
> `(2.14b)`:=let('tau[mu]=matrix([
  [-u[mu]+alpha[mu], beta[mu]-I*gamma[mu]],
  [beta[mu]+I*gamma[mu], -u[mu]-alpha[mu]]
  ])');
```

$$(2.14b) := \tau_{\mu} = \begin{bmatrix} -u_{\mu} + \alpha_{\mu} & \beta_{\mu} - I\,\gamma_{\mu} \\ \beta_{\mu} + I\,\gamma_{\mu} & -u_{\mu} - \alpha_{\mu} \end{bmatrix}$$

The Levi-Civita object (fundamental spinor)

```
> epsilon:=array(antisymmetric,1..2,1..2):
  epsilon[1,2]:=1:
> matrix(2,2,(i,j)->epsilon[i,j]);
```

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Time reversed basis

```
> Tau:=array(-4..4):
> let('Tau[-rho]=epsilon&*transpose(htranspose(Sigma[-rho]))&*eps
  ilon',
  simplify,real);
```

$$T_{-\rho} = (\varepsilon\ \&*\ (\Sigma_{-\rho}{}^{+})^{T})\ \&*\ \varepsilon$$

```
> vector(dimension,rho->evalm(Tau[-rho]));
```

$$\left[\begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2}I\sqrt{2} \\ \frac{1}{2}I\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix}\right]$$

```
> let('Tau[rho]=epsilon&*transpose(htranspose(Sigma[rho]))&*epsil
  on',
```

```
simplify,real);
```

$$T_\rho = (\varepsilon \,\&*\, (\Sigma_\rho^{+})^{T}) \,\&*\, \varepsilon$$

```
> vector(dimension,rho->evalm(Tau[rho]));
```

$$\left[\begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 \\ 0 & \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} I \sqrt{2} \\ -\frac{1}{2} I \sqrt{2} & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}\sqrt{2} & 0 \\ 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix}\right]$$

tau is also the time-reversal of sigma

Note: conjugate does not map into listlist or arrays, but transpose(htranspose(...)) is the same as the conjugate of a matrix.

```
> `(2.11a)`:=IS('tau[-rho]=epsilon&*transpose(htranspose(sigma[-r
  ho]))&*epsilon',
  simplify,real);
```

true

$$(2.11a) := \tau_{-\rho} = (\varepsilon \,\&*\, (\sigma_{-\rho}^{+})^{T}) \,\&*\, \varepsilon$$

```
> `(2.11b)`:=IS('tau[rho]=epsilon&*transpose(htranspose(sigma[rho
  ]))&*epsilon',
  simplify,real,evalm);
```

true

$$(2.11b) := \tau_{\rho} = (\varepsilon \,\&*\, (\sigma_{\rho}^{+})^{T}) \,\&*\, \varepsilon$$

Identity matrix

```
> sigma[0]:=evalm(2^(1/2)*Sigma[4]);
```

$$\sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Bergmann's "useful identities"

Note: The operator x->equal(lhs(x),rhs(x)) is required to compare matrices.

```
> `(2.18a)`:=IS('sigma[rho]&*tau[-rho]=-4*sigma[0]',
  simplify,evalm);
```

true

$$(2.18a) := \sigma_\rho \,\&*\, \tau_{-\rho} = -4\, \sigma_0$$

```
> `(2.18b)`:=IS('tau[rho] &* sigma[-rho]=-4*sigma[0]',
  simplify,evalm);
```

$$true$$

$$(2.18b) := \tau_{\rho}\ \&*\ \sigma_{-\rho} = -4\ \sigma_{0}$$

```
> `(2.18c)`:=IS('sigma[rho]&*sigma[-rho]=-2*sigma[0]',
  simplify,evalm);
```

$$true$$

$$(2.18c) := \sigma_{\rho}\ \&*\ \sigma_{-\rho} = -2\ \sigma_{0}$$

For an arbitrary matrix we have

```
> A:=matrix([[A11,A12],[A21,A22]]);
```

$$A := \begin{bmatrix} A11 & A12 \\ A21 & A22 \end{bmatrix}$$

```
> `(2.18d)`:=IS('sigma[rho]&*A&*tau[-rho]=
  -2*trace(A)*sigma[0]',
  simplify,evalm);
```

$$true$$

$$(2.18d) := (\sigma_{\rho}\ \&*\ A)\ \&*\ \tau_{-\rho} = -2\ \mathrm{trace}(A)\ \sigma_{0}$$

Spin-affine connection associated with the Levi-Civita connection.

```
> `(4.5)`:='Omega[rho]=-1/4*tau[mu]&*der(x[-rho],Gamma)(sigma[-mu
  ])';
```

$$(4.5) := \Omega_{\rho} = -\left(\left(\frac{1}{4}\tau_{\mu}\right)\&*\ (\mathrm{D}_{\rho}{}^{\Gamma})(\sigma_{-\mu})\right)$$

```
> let(`(4.5)`,simplify,evalm);
```

$$\Omega_{\rho} = -\left(\left(\frac{1}{4}\tau_{\mu}\right)\&*\left(\left(\frac{\partial}{\partial x_{-\rho}}\sigma_{-\mu}\right) + \Gamma_{-\mu,\ j',\ \rho}\ \sigma_{-j'}\right)\right)$$

It's trace and the trace of its Hemitean conjugate is zero.

```
> IS('trace(Omega[rho])=0',simplify);
```

$$true$$

$$\mathrm{trace}(\Omega_{\rho}) = 0$$

```
> IS('trace(htranspose(Omega[rho]))=0',simplify,real);
```

$$true$$

$$\mathrm{trace}(\Omega_{\rho}{}^{+}) = 0$$

Check that the covariant derivative of the spin matrices vanish.

```
> `(4.4)`:=IS(matrix(2,2,0)=
  der(x[-rho],Gamma)(sigma[-mu]) -
  ('htranspose'(Omega[rho])&*sigma[-mu]) -
  (sigma[-mu]&*Omega[rho]),
  simplify,real,evalm);
```

$$true$$

$$(4.4) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \left(\frac{\partial}{\partial x_{-\rho}} \sigma_{-\mu}\right) + \Gamma_{-\mu,\ j',\ \rho}\, \sigma_{-j'} - (\Omega_{\rho}^{\ +} \,\&*\, \sigma_{-\mu}) - (\sigma_{-\mu} \,\&*\, \Omega_{\rho})$$

Spin curvature

```
> `(4.6)`:='P[iota,kappa]=2*alt[iota,kappa](der(x[-kappa])(Omega[
  iota]))+2*alt[iota,kappa](Omega[kappa]&*Omega[iota])';
```

$$(4.6) := P_{\iota,\ \kappa} = 2\ alt_{\iota,\ \kappa}(\mathrm{D}_{\kappa}(\Omega_{\iota})) + 2\ alt_{\iota,\ \kappa}(\Omega_{\kappa} \,\&*\, \Omega_{\iota})$$

```
> infolevel[let]:=0:
  let(`(4.6)`,simplify);
```

$$P_{\iota,\ \kappa} = \left(\frac{\partial}{\partial x_{-\kappa}} \Omega_{\iota}\right) - \left(\frac{\partial}{\partial x_{-\iota}} \Omega_{\kappa}\right) + (\Omega_{\kappa} \,\&*\, \Omega_{\iota}) - (\Omega_{\iota} \,\&*\, \Omega_{\kappa})$$

Special routine for faster matrix comparison

```
> testm:=proc(exp)
  option remember;
    local i,j, R;
    R:=NULL;
    for i to 2 do
      for j to 2 do
        R:=R,testeq(
         simplify(real(
            lhs(exp)[i,j]-rhs(exp)[i,j]
         ))=0
        );
  #print([R][-1]);
      od
    od;
    convert([R],`and`)
  end:
```

Relationship between Riemann curvature and spin curvature

```
> infolevel[IS]:=0:
`(4.7a)`:=IS('R[iota,kappa,lambda,mu]*sigma[-lambda]-htranspose
(P[iota,kappa])&*sigma[mu]-sigma[mu]&*P[iota,kappa]'=matrix(2,2
,0),
testm);
```

$$true$$

$$(4.7a) := R_{\iota,\kappa,\lambda,\mu}\,\sigma_{-\lambda} - (P_{\iota,\kappa}{}^{+}\ \&*\ \sigma_{\mu}) - (\sigma_{\mu}\ \&*\ P_{\iota,\kappa}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

```
> infolevel[IS]:=0:
IS('R[iota,kappa,lambda,mu]=1/4*trace(htranspose(P[iota,kappa])
&*(sigma[lambda]&*tau[mu]-sigma[mu]&*tau[lambda])+(tau[mu]&*sig
ma[lambda]-tau[lambda]&*sigma[mu])&*P[iota,kappa])',
simplify,real);
```

$$true$$

$$R_{\iota,\kappa,\lambda,\mu} = \frac{1}{4}\,\mathrm{trace}((P_{\iota,\kappa}{}^{+}\ \&*\ ((\sigma_{\lambda}\ \&*\ \tau_{\mu}) - (\sigma_{\mu}\ \&*\ \tau_{\lambda}))) + (((\tau_{\mu}\ \&*\ \sigma_{\lambda}) - (\tau_{\lambda}\ \&*\ \sigma_{\mu}))\ \&*\ P_{\iota,\kappa}))$$

```
> `(4.10a)`:=IS('P[iota,kappa]=1/4*tau[-lambda]&*sigma[-mu]*R[iot
a,kappa,lambda,mu]',
testm);
```

$$true$$

$$(4.10a) := P_{\iota,\kappa} = \left(\left(\frac{1}{4}\,\tau_{-\lambda}\right)\&*\ \sigma_{-\mu}\right) R_{\iota,\kappa,\lambda,\mu}$$

```
> infolevel[IS]:=0:
IS('trace(P[iota,kappa])=0',simplify);
```

$$true$$

$$\mathrm{trace}(P_{\iota,\kappa}) = 0$$

The relationship between the spin curvature and the Ricci curvature: they are the same thing!

```
> infolevel[IS]:=0:
`(4.7b)`:=IS('R0[iota,kappa,lambda,mu]*sigma[-lambda]=-htranspo
se(P[iota,kappa])&*sigma[mu]-sigma[mu]&*P[iota,kappa]',
testm);
```

$$true$$

$$(4.7b) := R0_{\iota,\kappa,\lambda,\mu}\,\sigma_{-\lambda} = -(P_{\iota,\kappa}{}^{+}\ \&*\ \sigma_{\mu}) - (\sigma_{\mu}\ \&*\ P_{\iota,\kappa})$$

```
> infolevel[IS]:=0:
`(4.10b)`:=IS('P[iota,kappa]=-1/4*tau[-lambda]&*sigma[-mu]*R0[i
ota,kappa,lambda,mu]',
```

```
testm);
```

$$true$$

$$(4.10b) := P_{\iota,\,\kappa} = -\left(\left(\frac{1}{4}\,\tau_{-\lambda}\right)\&*\,\sigma_{-\mu}\right) R0_{\iota,\,\kappa,\,\lambda,\,\mu}$$

Spin-affine connection associated with the full affine connection
(suppose we replace Levi-Civita connection Γ with the affine connection Γ0)

```
> `(4.5x)`:='Omega2[rho]=-1/4*tau[mu]&*der(x[-rho],Gamma0)(sigma[
  -mu])';
```

$$(4.5x) := \Omega 2_{\rho} = -\left(\left(\frac{1}{4}\,\tau_{\mu}\right)\&*\,(D_{\rho}^{\Gamma 0})(\sigma_{-\mu})\right)$$

```
> let(`(4.5x)`,simplify,evalm);
```

$$\Omega 2_{\rho} = -\left(\left(\frac{1}{4}\,\tau_{\mu}\right)\&*\left(\left(\frac{\partial}{\partial x_{-\rho}}\,\sigma_{-\mu}\right) + \Gamma 0_{-\mu,\,j',\,\rho}\,\sigma_{-j'}\right)\right)$$

It's trace and the trace of its Hemitean conjugate is zero.

```
> IS('trace(Omega2[rho])=0',simplify);
```

$$true$$

$$\mathrm{trace}(\Omega 2_{\rho}) = 0$$

```
> IS('trace(htranspose(Omega2[rho]))=0',simplify,real);
```

$$true$$

$$\mathrm{trace}(\Omega 2_{\rho}^{+}) = 0$$

Check that the covariant derivative of the spin matrices vanish.

```
> `(4.4x)`:=IS(matrix(2,2,0)=
  der(x[-rho],Gamma0)(sigma[-mu]) -
  ('htranspose'(Omega2[rho])&*sigma[-mu]) -
  (sigma[-mu]&*Omega2[rho]),
  simplify,real,evalm);
```

$$true$$

$$(4.4x) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \left(\frac{\partial}{\partial x_{-\rho}}\,\sigma_{-\mu}\right) + \Gamma 0_{-\mu,\,j',\,\rho}\,\sigma_{-j'} - (\Omega 2_{\rho}^{+}\,\&*\,\sigma_{-\mu}) - (\sigma_{-\mu}\,\&*\,\Omega 2_{\rho})$$

Spin curvature

```
> `(4.6x)`:='P0[iota,kappa]=2*alt[iota,kappa](der(x[-kappa])(Omega2[iota]))+2*alt[iota,kappa](Omega2[kappa]&*Omega2[iota])';
```

$$(4.6x) := P0_{\iota,\kappa} = 2\, alt_{\iota,\kappa}(D_\kappa(\Omega 2_\iota)) + 2\, alt_{\iota,\kappa}(\Omega 2_\kappa \,\&{*}\, \Omega 2_\iota)$$

```
> infolevel[let]:=0:
let(`(4.6x)`,simplify);
```

$$P0_{\iota,\kappa} = \left(\frac{\partial}{\partial x_{-\kappa}} \Omega 2_\iota\right) - \left(\frac{\partial}{\partial x_{-\iota}} \Omega 2_\kappa\right) + (\Omega 2_\kappa \,\&{*}\, \Omega 2_\iota) - (\Omega 2_\iota \,\&{*}\, \Omega 2_\kappa)$$

```
> infolevel[IS]:=1:
`(4.7x)`:=IS('S[iota,kappa,lambda,mu]*sigma[-lambda]-htranspose(P0[iota,kappa])&*sigma[mu]-sigma[mu]&*P0[iota,kappa]'=matrix(2,2,0),
simplify);
```

true

$$(4.7x) := S_{\iota,\kappa,\lambda,\mu}\, \sigma_{-\lambda} - (P0_{\iota,\kappa}^{+} \,\&{*}\, \sigma_\mu) - (\sigma_\mu \,\&{*}\, P0_{\iota,\kappa}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

<u>But as we saw above</u> $S_{i,k,l,m} = 0$ **<u>so the</u>** ***P0*** **<u>spin curvature is zero!</u>**

In other words, if we replace the Levi-Civita connection with the affine connection, then all of the curvature is expressed as "spin curvature".

```
> infolevel[IS]:=0:
IS('S[iota,kappa,lambda,mu]=1/4*trace(htranspose(P0[iota,kappa])&*(sigma[lambda]&*tau[mu]-sigma[mu]&*tau[lambda])+(tau[mu]&*sigma[lambda]-tau[lambda]&*sigma[mu])&*P0[iota,kappa])',
simplify,real);
```

true

$$S_{\iota,\kappa,\lambda,\mu} = \frac{1}{4}\,\mathrm{trace}((P0_{\iota,\kappa}^{+} \,\&{*}\, ((\sigma_\lambda \,\&{*}\, \tau_\mu) - (\sigma_\mu \,\&{*}\, \tau_\lambda))) + (((\tau_\mu \,\&{*}\, \sigma_\lambda) - (\tau_\lambda \,\&{*}\, \sigma_\mu)) \,\&{*}\, P0_{\iota,\kappa}))$$

Shipov's Hermitean matrix basis.

```
> sigma2:=array(-4..4,1..2,1..2):
> infolevel[let]:=0:
`(6.9)`:=let('sigma2[-a,A,B]=2^(-1/2)*[
[[1,0],[0,-1]],
[[0,1],[1,0]],
[[0,-I],[I,0]],
[[1,0],[0,1]]
```

```
]
[a,A,B]'):
```

```
> 'matrix(2,2,(A,B)->2^(1/2)*sigma2[-a,A,B])'$a=1..4;
```

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> IS('sigma2[-a,A,B]=Sigma[-a][A,B]');
```

$$true$$

$$\sigma 2_{-a,\,A,\,B} = \Sigma_{-a_{A,\,B}}$$

Shipov's equality (page 225)

```
> IS('eta[a,b]*Sigma[-a][A,`B'`]*Sigma[-b][C,`D'`]=epsilon[A,C]*e
psilon[`B'`,`D'`]');
```

$$true$$

$$\eta_{a,\,b}\,\Sigma_{-a_{A,\,B'}}\,\Sigma_{-b_{C,\,D'}} = \varepsilon_{A,\,C}\,\varepsilon_{B',\,D'}$$

```
> IS(2^(1/2)*Sigma[-a]*E[-mu,a]=sigma[-mu],simplify);
```

$$true$$

$$\sqrt{2}\,\Sigma_{-a}\,E_{-\mu,\,a} = \sigma_{-\mu}$$

```
> IS('2^(1/2)*Sigma[a]*e[-a,mu]=sigma[mu]',simplify);
```

$$true$$

$$\sqrt{2}\,\Sigma_{a}\,e_{-a,\,\mu} = \sigma_{\mu}$$

```
> Epsilon:=array(antisymmetric,-2..-1,-2..-1):
Epsilon[-1,-2]:=1:
```

```
> matrix(2,2,(i,j)->Epsilon[-i,-j]);
```

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

```
> infolevel[IS]:=0:
`(6.34)`:=IS(G[i,j]=1/2*Epsilon[-A,-C]*Epsilon[-`B'`,-`D'`]*sig
ma[i][A,`B'`]*sigma[j][C,`D'`],simplify);
```

$$true$$

$$(6.34) := G_{i,\,j} = \frac{1}{2}\,\mathrm{E}_{-A,\,-C}\,\mathrm{E}_{-B',\,-D'}\,\sigma_{i_{A,\,B'}}\,\sigma_{j_{C,\,D'}}$$

```
>
>
```

Save everything for the next time

```
> save `bergmann4d7.m`;
>
```