Quaternions, Torsion and the Physical Vacuum; Theories of M. Sachs and G. Shipov Compared

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Abstract. Of several developments of unified field theories in the spirit of Einstein's original objective of a fully geometric description of all classical fields as well as quantum mechanics, two are particularly noteworthy. The works of Mendel Sachs and Gennady Shipov stand apart as major life works comprising tens of papers, several monographs and decades of effort. Direct comparison of these theories is hampered however by differences in notation and conceptual viewpoint. Despite these differences, there are many parallels between the fundamental mathematical structures appearing in each. In this paper we discuss the main tenets of the two approaches and demonstrate that they both give rise to a factorization of the invariant interval of general relativity.

1 Introduction

The theories reviewed in this paper represent a return to the ideas initiated by Einstein after the development of general relativity. After briefly introducing both theories we develop the representations used by each in such as way as to demonstrate their underlying similarity and how each leads to a factorization of the general invariant space-time line element.

In his book, General Relativity and Matter[1] Mendel Sachs presents a unified field theory incorporating gravitation, electromagnetism, nuclear interactions and the inertial properties of matter. In a later book, Quantum Mechanics from General Relativity[2] Sachs extends the formalism of general relativity in the manner originally envisioned by Einstein to obtain a general theory of matter including those properties of matter that are now usually described by quantum mechanics. To achieve this unification Sachs writes the field equations of general relativity in a factored form having a similar relationship to the usual field equations of general relativity that the Dirac equation has to the Klein-Gordon equation in relativistic quantum mechanics.

Factoring the field equations involves introducing a generalization of Riemann geometry that admits coordinate transformations involving all 16 parameters of the Einstein group rather than the usual 10 parameters of the Poincare group. These extra parameters represent spin degrees of freedom. Sachs expresses this in terms of the algebra of spinors and quaternions. Applying this same factorization to Maxwell's equations leads to an explanation the of Lamb shift without involving quantum field theory.

Exact solution of the field equations corresponding to a ground state of bound particle-antiparticle pairs suggests a picture of the physical vacuum quite different than the virtual particle sea of contemporary relativistic quantum mechanics. Instead of annihilating, ground state particle-antiparticle pairs constitute a ubiquitous very weakly interacting background which provides an alternate physical interpretation of phenomena such as anomalous scattering and magnetic moments that are well described numerically by contemporary relativistic quantum mechanics but lack an intuitive physical interpretation in that formalism.

Gennady Shipov, in his book "A Theory of Physical Vacuum"[3] also presents a unified field theory with conclusions very similar to those of Mendel Sachs. Shipov's programme involves a completely geometric representation of the field equations of general relativity as equivalent to the structural equations of A_4 geometry[8]. Gravitation and the inertial properties of matter in non-inertial frames of reference are described in terms of the contorsion part of the general affine connection of A_4 while a generalization of electromagnetism is derived from the Christoffel part of this connection.

Solutions to the structural equations for the situation corresponding to anti-particle particle pairs bound by the generalized electromagnetic interaction yield the same picture of the physical vacuum as proposed by Sachs. Shipov's theory achieves William Clifford's vision[4] that preceded Einstein's general relativity by more than 30 years, of a representation of the material world entirely in terms of the curved and twisted geometry of space itself.

It is remarkable that in addition to the description of gravitational, electromagnetic and nuclear interactions that are well known in physics, Shipov's theory also admits solutions involving only the torsion of space. Shipov proposes some novel interpretations and potential applications of this fact that are very controversial.

From the surface resemblance seen in the above comparison one is led to consider the possibility that the two theories may be linked at a fundamental level. However differences not only in notation but in the choice of affine connections and geometry act as barriers to direct comparison. We have undertaken a research programme directed towards construction of a bridge between the formalisms and determination of their relationship to each other. The objectives of this paper which contains some early results from this effort include: familiarizing readers with the existence of the two theories; making available a readable derivation of the spinor affine connection used by Sachs and a parallel development for Shipov's connection; identifying the number of spin degrees of freedom retained in each theory's metric factorization and as a result demonstrating another parallel between the approaches.

2 Spinors

Fundamental to the development by Sachs is the application of the spinor representation of space-time. Cartan[5] introduced the spinor as an irreducible representation of the proper Lorentz group of special relativity. The splitting of the four dimensional Riemannian space into a direct product of two spinor spaces was first introduced by Van Der Waerden and Infield[6] with the introduction of spinor analysis. This application of spinors was further developed by Bergmann[7] and many others, eventually taking a form that is today recognized as a theory of connections on a complex valued fibre-bundle.

Recall that the covariant derivative of a vector is given by

$$b_{\mu;\rho} = b_{\mu,\rho} - \{^{\nu}_{\mu\rho}\} b_\nu \tag{1}$$

where $\{^{\nu}_{\mu\rho}\}$ is the affine connection

The covariant derivative of a two-component spinor ψ is represented by means of a set of fundamental 2nd rank spinor fields called the spin-affine connection Ω_ρ. Ω_ρ plays the same role as the tetrad field in the more well known tetrad tensor field formalism.

$$\psi_{;\rho} = \psi_{,\rho} + \Omega_\rho \psi \tag{2}$$

A mapping is needed between the space of spinors and tensors that allows us to represent tensors with full compatibility between actions carried out with tensor objects and the results one would obtain on first mapping into spinor space and then carrying out these same actions. We shall find that such a mapping can be found through a tetrad of fundamental fields q^μ which are mixed tensor/spinor objects that take a covariant tensor b_μ into a 2nd rank spinor β by means of the simple mapping

$$\beta = b_\mu q^\mu \tag{3}$$

Our compatibility requirement when applied to the action of the covariant derivative then requires that we obtain the same value on differentiation followed by mapping as mapping followed by differentiation. Hence we need to find tetrad fields that obey the equality

$$\beta_{;\rho} = (b_\mu q^\mu)_{;\rho} = b_{\mu;\rho} q^\mu \tag{4}$$

Applying Liebnitz's rule we obtain the requirement

$$b_{\mu;\rho} q^\mu + b_\mu q^\mu_{;\rho} = b_{\mu;\rho} q^\mu \tag{5}$$

$$b_\mu q^\mu_{;\rho} = 0 \tag{6}$$

which can only be fulfilled if $q^\mu_{;\rho} = 0$. Thus, a necessary condition on the existence of a compatible spinor representation of a tensor is the existence of a tetrad field with this property. Given such a tetrad field, we need now also obtain the spinor affine connection Ω_ρ that realizes this compatibility.

Before proceeding we will need to also introduce the covariant second rank fundamental spinor

$$\varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{7}$$

(the contravariant form being given by ε^T) which plays a role for spinors similar to the fundamental metric tensor for the definition of an inner product and in the raising and lowering of spinor indices.

The covariant derivative of the fundamental spinor is obtained again by a correspondence principle. Given a spinor invariant formed via the metric property of this spinor, we require that the covariant derivative of this new spinor valued object to behave appropriately in analogy with the covariant derivative of a scalar field:

$$(\psi\varepsilon\psi^T)_{;\rho} = 0. \tag{8}$$

Carrying out the covariant derivative we obtain

$$\psi_{;\rho}\varepsilon\psi^T + \psi\varepsilon\psi^T_{;\rho} + \psi\varepsilon_{;\rho}\psi^T = 0 \tag{9}$$

Thanks to the antisymmetric nature of ε it can be easily shown that all components of the first two terms cancel, leaving the requirement that $\varepsilon_{;\rho} = 0$ since ψ is arbitrary. It was shown by Bergmann that the vanishing of the covariant derivatives of the tetrad quaternion field and of the metric spinor are sufficient conditions to restrict the allowable solutions for the spin-affine connection to a unique solution in the case of a spinor space based upon the restriction of unimodular spinor transformations.

By definition the fundamental quaternion fields q^μ transforms as a 2nd rank covariant spinor and as a tensor with respect to the tensor index μ. That is, we have that

$$q^{\mu'} = (\partial x^{\mu'}/\partial x^\mu)(b^\dagger)^{-1}q^\mu b^{-1}.$$

Applying the Leibnitz rule, the covariant derivative of these fields can be written as follows if we assume that the connection for tensor objects takes the form appropriate for a V_4 space which is given by the Christoffel symbols.

$$q^\mu_{;\rho} = \partial_\rho q^\mu + \{{}^{\mu}_{\tau\rho}\}q^\tau - \Omega^\dagger_\rho q^\mu - q^\mu\Omega_\rho \tag{10}$$

or

$$\tilde{q}^\mu_{;\rho} = \partial_\rho \tilde{q}^\mu + \{{}^{\mu}_{\tau\rho}\}\tilde{q}^\tau + \Omega_\rho\tilde{q}^\mu + \tilde{q}^\mu\Omega^\dagger_\rho$$

where $\{{}^{\mu}_{\tau\rho}\}$ are the Christoffel symbols. $\tilde{q}$ denotes the time-reversed quaternion field (so named as the action that results is reversal of the sign of the x_0 component in x)

$$\tilde{q}^\mu = \varepsilon(q^\mu)^*\varepsilon \tag{11}$$

where * denotes the complex conjugate.

The fundamental spinor plays the role of the fundamental metric tensor in the raising/lowering of spinor indices and in the construction of the inner product and magnitude of spinors. Bergmann requires that the covariant derivative of the fundamental quaternion fields vanish, i.e. that transport of these fields and the fundamental spinor ε from one point in space to an infinitesimally near point are both globally parallel.

$$q^\mu_{;\rho} = 0$$

$$\varepsilon_{;\rho} = 0$$

As shown by Bergmann, on obtaining a solution of these equations the resulting spinor-affine connection is uniquely obtained as

$$\Omega_\rho = \frac{1}{4}\left(\tilde{q}^\mu_{,\rho} + \{{}^{\mu}_{\tau\rho}\}\tilde{q}^\tau\right)q_\mu \tag{12}$$

Sachs introduces the new notion of an algebraic structure for Bergmann's tetrad fields. He shows that the q^μ can be interpreted as a quaternion valued four vector and as such admits the manipulations of quaternion algebra.

2.1 Clifford Algebra

The real-valued quaternion algebra is the even sub-algebra of the Clifford algebra of 3-dimensional space $Cl(3)$. $Cl(3)$ is isomorphic to the algebra of 2x2 complex matrices and has also been called complex-valued quaternion algebra. The matrix representation of the basis of $Cl(3)$ consists of 8 matrices:

the identity matrix (rank 0),

$$\mathbf{1} = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{13}$$

Pauli matrices (rank 1),

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

the (rank 2) products

$$\sigma_{12} = \sigma_1\sigma_2, \quad \sigma_{13} = \sigma_1\sigma_3, \quad \sigma_{23} = \sigma_2\sigma_3 \tag{14}$$

$$\mathbf{i} = -\sigma_{23} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \quad \mathbf{j} = -\sigma_{31} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{k} = -\sigma_{12} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \tag{15}$$

and the pseudo-scalar (rank 3)

$$\sigma_{123} = \sigma_1\sigma_2\sigma_3 = i\sigma_0 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \tag{16}$$

The even rank elements of $Cl(3)$ constitute the basis for the quaternion

$$Q = q_0 1 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} = \begin{bmatrix} q_0 - iq_3 & -iq_1 - q_2 \\ -iq_1 + q_2 & q_0 + iq_3 \end{bmatrix} \tag{17}$$

Since the second rank spinor fields have the form of 2x2 Hermitean matrices they may be represented as quaternions. As will be seen, Sachs exploits the associated algebraic structure to obtain his factorization of the metric.

3 Sachs' Factorization

Sachs observes that the structure of the 10 parameter Poincare group, which includes translations, rotations and reflections, is represented in the Riemann geometry of conventional general relativity by the real-valued symmetric metric tensor $g_{\mu\nu}$. But reflection symmetry is not required by any of the postulates of general relativity. If operations of reflection are removed from the Poincare group, the result is a 16 parameter group that Sachs calls the Einstein group. $g_{\mu\nu}$ does not provide a complete representation of this group. But a faithful irreducible representation can be found in terms of the fundamental quaternion field. Hence the metric tensor can be written in the symmetric factored form

$$\sigma_0 g^{\mu\nu} = -q^{(\mu}\tilde{q}^{\nu)} \tag{18}$$

where the products of the field tensor components are understood as quaternion products. Now we can write the linear invariant infinitesimal line element as the quaternion differential

$$ds = q^{\mu} dx_{\mu} \quad d\tilde{s} = \tilde{q}^{\mu} dx_{\mu} \tag{19}$$

Thus ds is a quaternion-valued scalar invariant. In contrast to the conventional formulation, this invariant no longer has any ambiguity of sign. It is invariant with respect to translations in space but has internal spin degrees of freedom. Sachs also defines the quaternion conjugate or time-reversed quaternion field $\tilde{q}^{\mu}$ from q^{μ}

$$d\tilde{s} = \tilde{q}^{\mu} dx_{\mu} \tag{20}$$

Their product

$$ds^2 = ds\, d\tilde{s} = \sigma_0 g^{\mu\nu} dx_{\mu} dx_{\nu} \tag{21}$$

is the ordinary quadratic real-valued line element of Riemann space which is invariant with respect to changes in both spinor coordinates and translations. This factorization makes apparent "spin" degrees of freedom that are usually hidden.

Sachs does not address the important questions of the number of degrees of freedom in the quaternion field that are preserved in the invariant interval differentials. By finding the rank of the Jacobian of eight differential components in ds and $d\tilde{s}$ with respect to the sixteen coefficients in the quaternion field we find that there are exactly four degrees of freedom. This is suggestive of the form of the intrinsic spin four vector.

4 Shipov's Tetrads

Shipov concerns himself from the beginning of his development with associating angular reference frames to point-size entities. To accomplish this he applies the concept of tetrads. We will briefly introduce this approach in this section.

The method of tetrads or vierbien in the tensor analysis used in the early work on general relativity and unified field theories does not lead naturally to the full irreducible representation of the properties of higher-order geometry. For example, in tensor notation, the Riemann metric is written as follows:

$$g_{\mu\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu} \tag{22}$$

where the tetrad e^a_{μ} consists of four linearly independent covariant vector fields which provide a local pseudo-Euclidean coordinate system at each point. We also have contravariant vector fields such that

$$e^a_{\mu} e^{\mu}_b = \delta^a_b = \{ {}^{1\ a=b}_{0\ a\neq b} \tag{23}$$

and

$$e^a_{\mu} e^{\nu}_a = \delta^{\nu}_{\mu} \tag{24}$$

We use Greek letters such as μ, ν etc. to denote tensor indices and Latin letters such as a, b etc. to denote "tetrad" indices. Note that raising and lowering tetrad indices is done via the Minkowski metric $\eta_{ab} = \eta^{ab}$

of the local coordinate system, while tensor indices involve the symmetric Riemann metric $g_{\mu\iota}g^{\iota\nu} = \delta^{\nu}_{\mu}$. The invariant differential interval ds is written

$$ds^2 = g_{\mu\nu}dx_{\mu}dx_{\nu} \tag{25}$$

The tetrad fields map a tensor into set of tetrad scalars

$$\beta_a = b_{\mu}e^{\mu}_{a} \tag{26}$$

a is a "dead" index (Schouten). We may now consider the covariant derivative of these tetrad scalars

$$\beta_{a;\rho} = \beta_{a,\rho} + T^{b}_{a\rho}\beta_b \tag{27}$$

$T^{b}_{a\rho}$ are Ricci rotation coefficients

As in the discussion of the covariant derivative of spinors above, we may ask when is the covariant derivative of the tetrad scalars compatible with tensor differentiation.

$$\beta_{a;\rho} = (b_{\mu}e^{\mu}_{a})_{;\rho} = b_{\mu;\rho}e^{\mu}_{a} \tag{28}$$

The analogous necessary condition is

$$e^{\mu}_{a;\rho} = 0 \tag{29}$$

which leads to the definition of the Ricci rotation coefficients

$$T^{b}_{a\rho} = \left(e^{\mu}_{a,\rho} + \{{}^{\mu}_{\tau\rho}\}e^{\tau}_{a}\right)e^{b}_{\mu} \tag{30}$$

where the inverse tetrad is defined by

$$e^{\mu}_{a}e^{a}_{\nu} = \{{}^{1,\mu=\nu.}_{0,\mu\neq\nu} \tag{31}$$

Hence identical prescriptions are used by Shipov and Sachs in the derivation of the connection for their respective geometries.

4.1 Metric Factorization

Using the tetrad bases we may form the four linear scalar invariants

$$ds^a = e^{a}_{\mu}dx^{\mu} \quad ds_a = e^{\mu}_{a}dx_{\mu} \tag{32}$$

and from these usual quadratic Riemann metric

$$ds^2 = ds^a ds_b = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{33}$$

Thus the tetrad fields allow a factorization of the invariant interval without sign ambiguity but within the context of Shipov's formalism and without introduction of the spinor/quaternion calculus of Sachs. Now since ds_a and ds^a are linearly related through a raising operation (by the Minkowski metric) by virtue of the construction given above, these represent only four degrees of freedom. Hence the Shipov differential invariants comprise the same number of spin degrees of freedom as the Sachs invariants. In the Shipov construction, the raising/lowering operation in the internal Minkowski tangent space at each point in his A4 geometry is the parallel of the process of quaternion conjugation in Sachs' case.

5 Conclusion

The resurrection of unified field theory, as originally envisioned by Einstein, Cartan and many others following the development of general relativity, represents a clear alternative to the collection of phenomenological and mathematical procedures loosely referred to as the "standard model".

The approach to higher order geometry required to express absolute parallelism and exemplified by the spinor formulation obviate the need to appeal to physically unintuitive notions such as strings in 10 dimensional space.

At this stage of our research programme we conclude that there is a deep similarity between Sachs' spinor and quaternion development and Shipov's tetrad based formalism, not only in general perspectives but at the level of metric factorization. On the other hand, there are such pronounced differences in notation and geometric formalism that further study will be required to determine whether or not the similarity extends to an isomorphism. It our intent to pursue this investigation to such ends.

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